

# Charm quark mass

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## Abstract

We report on the result for the charm quark mass as obtained from our lattice QCD computation in the quenched approximation. Our result is  $m_c^{\overline{\text{MS}}}(m_c) = 1.26(4)(12)$  GeV.

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## 1 Introduction

### 1.1 Status of the charm quark mass calculations

During the last few years a great effort has been devoted to the precise determination of the quark masses. Recent reviews about the present situation concerning the computation of the light quark masses can be found in refs. [1, 2]. As for the heavies, most of the studies done so far were focused on determining the  $b$ -quark mass. The charmed quark escapes the precision computation mainly because it is too heavy for the chiral perturbation theory to apply, and yet too light for the heavy quark expansion to set in.

A complete account of the presently available estimates of the charm quark mass value is given in the PDG review [3]. They estimate the charm quark mass to be in the range <sup>1</sup>

$$1.15 \text{ GeV} \leq m_c^{\overline{\text{MS}}}(m_c) \leq 1.35 \text{ GeV} . \quad (1)$$

Very recently, two new QCD sum rule computations of this quantity appeared [4, 5]. After improving the calculation of the moment QCD sum rules for the charmonium states, the new result of ref. [4] is  $m_c^{\overline{\text{MS}}}(m_c) = 1.23(9)$  GeV. Adopting quite a different QCD sum rule technique, in ref. [5]  $m_c^{\overline{\text{MS}}}(m_c) = 1.37(9)$  GeV was obtained.

On the side of the lattice QCD computations there were a few attempts to compute the charm quark.

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<sup>1</sup>Ref. [3] also contains a complete list of references.

- By combining the QCD sum rule methodology with the lattice QCD computation of the moments of the heavy-heavy vector current correlation function, the authors of ref. [6] obtained  $m_c^{\overline{\text{MS}}}(m_c) = 1.22(5)$  GeV. More complete discussion of the systematic uncertainties involved in their computation has been made in ref. [7]. Unfortunately, this (elegant) method has not been followed by the other lattice groups.
- From the lattice QCD with Wilson quarks and by using the Ward identities, in ref. [8] a much larger value has been obtained, namely  $m_c^{\overline{\text{MS}}}(m_c) = 1.71(3)(20)$  GeV. If we use their recent re-estimate of the mass renormalization constant [9], the above value gets down to  $m_c^{\overline{\text{MS}}}(m_c) = 1.50(3)(18)$  GeV.
- Finally, in ref. [10] the charm quark mass has been computed by adopting the Fermilab way of interpreting the lattice QCD beyond the lattice cut-off. The quoted result is  $m_c^{\overline{\text{MS}}}(m_c) = 1.33(8)$  GeV.

Since the actual situation with the lattice results is not clear, we decided to take advantage of the presently available non-perturbatively determined parameters, which are necessary for the complete elimination of the discretization effects that are linear in the lattice spacing (*i.e.* of  $\mathcal{O}(a)$ ), and to make an estimate of the charm quark mass from the data that we obtained by working with the (relatively) fine grained lattice ( $a \approx 0.07$  fm). Our result is

$$m_c^{\overline{\text{MS}}}(m_c) = 1.26(3)(12) \text{ GeV} . \quad (2)$$

## 1.2 Computation of the charm quark mass on the lattice

To compute the charm quark mass, we rely on the standard two strategies:

- Vector Ward identity (VWI)

$$m_Q^{(\text{VWI})}(\mu) = Z_m(\mu)m_Q^{(\text{VWI})} = Z_m(\mu)\frac{1}{2}\left(\frac{1}{\kappa_Q} - \frac{1}{\kappa_{cr}}\right) , \quad (3)$$

where the critical value of the Wilson hopping parameter ( $\kappa_{cr}$ ) corresponds to the chiral limit.<sup>2</sup> Since the lattice cutoff is finite and the charm quark mass is not negligible, it is highly important to improve the renormalization constant out of the chiral limit:

$$Z_m(\mu) = \frac{1}{Z_S^{(0)}(\mu)}(1 + b_m m_Q^{(\text{VWI})}) , \quad (4)$$

where the value of  $b_m = -b_S/2$ , has been determined non-perturbatively in ref. [11], and  $Z_S^{(0)}(\mu)$  is the renormalization constant of the scalar density computed in the chiral limit, which will be given in the next section.

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<sup>2</sup>Unless the physical units are explicitly displayed, all the quark masses mentioned in this section are assumed to be in lattice units.

- Axial-vector Ward identity (AWI)

$$m_Q^{(\text{AWI})}(\mu) + m_q^{(\text{AWI})}(\mu) = \overline{Z}_m(\mu) \frac{\langle \sum_{\vec{x}} \partial_4 A_4^{\text{I}}(x) P^\dagger(0) \rangle}{\langle \sum_{\vec{x}} P(x) P^\dagger(0) \rangle}, \quad (5)$$

where  $A_\mu = \bar{Q} \gamma_\mu \gamma_5 q$  and  $P = \bar{Q} \gamma_5 q$  are the axial vector current and the pseudoscalar density, respectively. For the full elimination of  $\mathcal{O}(a)$  effects, the bare lattice axial current is improved in the chiral limit as  $A_\mu^{\text{I}} = A_\mu + c_A \partial_\mu P$ , with  $c_A$  already determined non-perturbatively [12, 13]. We used the symmetric definition of the derivative, *e.g.*  $\partial_4 f = (f(t+1) - f(t-1))/2$ . The mass renormalization constant is equal to  $\overline{Z}_m(\mu) = Z_A/Z_P(\mu)$  and it is improved as

$$\overline{Z}_m(\mu) = \frac{Z_A^{(0)}}{Z_P^{(0)}(\mu)} \left( 1 + (b_A - b_P) \frac{m_Q^{(\text{VWI})} + m_q^{(\text{VWI})}}{2} \right). \quad (6)$$

Again, the value of  $(b_A - b_P)$  has been computed non-perturbatively [11, 12], whereas the value of  $\overline{Z}_m^{(0)}(\mu) = Z_A^{(0)}/Z_P^{(0)}(\mu)$ , will be provided in the next section.

## 2 Lattice details and results

### 2.1 A few details on the lattice computation

We work with two sets of lattice data, generated at  $\beta = 6.2$  by using the non-perturbatively improved Wilson action ( $c_{SW} = 1.614$  [13]). Each set contains 200 independent  $SU(3)$  gauge field configurations. The value of the critical parameter,  $\kappa_{cr}$ , is fixed by requiring the bare  $m_q^{(\text{AWI})} \rightarrow 0$ . The values of  $\kappa_{cr}$ , along with the values of the bare quark masses for the degenerate light quark combinations, as well as for the non-degenerate (heavy-light) ones, are given in tab. 1. Each  $m^{(\text{AWI})}$  and  $m^{(\text{VWI})}$  are obtained as

$$\begin{aligned} m^{(\text{AWI})} &= \frac{1}{2} \frac{\langle \sum_{\vec{x}} \partial_4 A_4^{\text{I}}(x) P^\dagger(0) \rangle}{\langle \sum_{\vec{x}} P(x) P^\dagger(0) \rangle}, \\ m^{(\text{VWI})} &= \frac{1}{4} \left( \frac{1}{\kappa_Q} + \frac{1}{\kappa_q} - \frac{2}{\kappa_{cr}} \right), \end{aligned} \quad (7)$$

where in the improved axial current,  $A_4^{\text{I}} = \bar{Q} \gamma_4 \gamma_5 q + c_A \partial_4 \bar{Q} \gamma_5 q$ , we used  $c_A = -0.04(1)$  [12, 13]. The statistical errors quoted in this work are obtained by using the standard jackknife procedure (with 5 configurations per jack). The quark masses listed in tab. 1 are the bare lattice ones, that we now need to renormalize. We next discuss the computation of the mass renormalization constants.

Lattice ( $\beta = 6.2$ )	$24^3 \times 64$ ( $\kappa_{cr} = 0.13583(5)$ )			$24^3 \times 48$ ( $\kappa_{cr} = 0.13577(3)$ )		
$\kappa_1 - \kappa_2$	$m^{(AWI)}$	$m^{(VWI)}$	$M_P$	$m^{(AWI)}$	$m^{(VWI)}$	$M_P$
0.1344–0.1344	0.0393(15)	0.0391(13)	0.305(2)	0.0399(14)	0.0376(9)	0.307(2)
0.1349–0.1349	0.0249(10)	0.0253(13)	0.244(2)	0.0241(12)	0.0237(9)	0.245(2)
0.1352–0.1352	0.0162(9)	0.0171(13)	0.200(3)	0.0155(11)	0.0154(9)	0.200(2)
0.125–0.1344	0.176(2)	0.179(1)	0.690(1)	0.172(2)	0.177(1)	0.693(2)
0.125–0.1349	0.167(2)	0.172(1)	0.672(2)	0.162(2)	0.170(1)	0.675(2)
0.125–0.1352	0.163(3)	0.168(1)	0.661(2)	0.158(3)	0.166(1)	0.663(2)
0.122–0.1344	0.221(3)	0.228(1)	0.786(1)	0.217(3)	0.226(1)	0.789(2)
0.122–0.1349	0.212(3)	0.221(1)	0.768(2)	0.209(3)	0.220(1)	0.771(2)
0.122–0.1352	0.207(3)	0.217(1)	0.757(2)	0.202(3)	0.215(1)	0.761(3)
0.119–0.1344	0.267(3)	0.280(1)	0.876(1)	0.261(3)	0.278(1)	0.878(2)
0.119–0.1349	0.258(3)	0.273(1)	0.858(2)	0.252(3)	0.271(1)	0.861(3)
0.119–0.1352	0.251(3)	0.269(1)	0.847(3)	0.248(4)	0.267(1)	0.851(4)

Table 1: Bare quark masses obtained by using the vector and the axial Ward identity on the lattice (see eq. (7)). The results are given in lattice units. On the subset of the configurations gathered on the lattice  $24^3 \times 64$ , in ref. [14] we computed the strange quark mass.

## 2.2 Mass renormalization constants

To evaluate the mass renormalization constants in the chiral limit we use the non-perturbative method [15] which allows one to compute these constants in the continuum RI/MOM scheme. The use of this method out of the chiral limit, however, requires the improvement of the off-shell quantities for which several new counterterms appear, each with a coefficient that is to be fixed non-perturbatively (for more details, see ref.[16]). For that reason, in this paper, we will employ the method to compute the renormalization constants in the chiral limit only [17]. To improve the renormalization constants out of the chiral limit, we will rely on the results of ref. [11, 12] in which the corresponding coefficients were computed non-perturbatively with the results,  $b_m = -0.69(1)$ ,  $b_A - b_P = 0.04$ , at  $\beta = 6.2$ .

We first focus on the mass renormalization constant,  $Z_m^{(0)}(\mu) = 1/Z_S^{(0)}(\mu)$ . To that end, we combine the quark propagators of 4 different values of  $\kappa_q$  in 10 different combinations (degenerate and non-degenerate in light quark mass), compute the amputated vertex function  $\Gamma_S(\kappa_i, \kappa_j; a\mu)$  and impose the standard RI/MOM renormalization condition <sup>3</sup>

$$\frac{1}{Z_S^{(0)}(\mu)} = \lim_{\kappa_{1,2} \rightarrow \kappa_{cr}} \frac{1}{Z_S(\kappa_1, \kappa_2; \mu)} = \lim_{\kappa_{1,2} \rightarrow \kappa_{cr}} \frac{\Gamma_S(\kappa_i, \kappa_j; p)}{Z_q^{1/2}(\kappa_i; p) Z_q^{1/2}(\kappa_j; p)} \Big|_{p^2=\mu^2}, \quad (8)$$

where  $Z_q(\mu)$  is the quark field renormalization constant which is easily extracted by imposing the vector Ward identity on the quark propagator. Such an obtained  $Z_m^{\text{RI/MOM}}(\mu) = 1/Z_S^{\text{RI/MOM}}(\mu)$  is then converted to the renormalization group invariant (RGI) form,  $Z_m^{\text{rgi}} = Z_m^{\text{RI/MOM}}(\mu)/c^{\text{RI/MOM}}(\mu)$ , by using the available mass anomalous dimension coefficients up to 4-loops encoded in the function  $c^{\text{RI/MOM}}(\mu)$ , computed in the same RI/MOM scheme [18]. <sup>4</sup> For consistency, we also use the 4-loop expression for the running coupling [19], and set  $n_F = 0$  with (quenched)  $\Lambda_{\text{QCD}}^{(n_F=0)} = 0.25 \text{ GeV}$  [20, 21]. A typical situation is shown in fig. 1(left). For every mass combination, we then extract  $Z_m^{\text{rgi}}(\mu)$  by fitting to a constant on the plateau  $1.0 \leq (a\mu)^2 \leq 1.8$ . After extrapolating in  $a\overline{m}_q = (m_1^{(\text{VWI})} + m_2^{(\text{VWI})})/2$  to the chiral limit, we get

$$Z_m^{(0)\text{rgi}} = 3.391(22). \quad (9)$$

For an easier comparison with the other determinations of this renormalization constant we express this result also in the  $\overline{\text{MS}}$  scheme:

$$Z_m^{(0)\overline{\text{MS}}}(2 \text{ GeV}) = 1.332(9). \quad (10)$$

As it can be observed from fig. 1(right), the extrapolation to the chiral limit in this case is very smooth.

Next, we discuss the computation of the second mass renormalization constant, *i.e.* the one needed to compute the quark mass by using the axial Ward identity ( $\overline{Z}_m^{(0)}(\mu) = Z_A^{(0)}/Z_P^{(0)}(\mu)$ ). We use the proposal of ref. [9] which, by judiciously combining the Ward identities, allows one to alleviate the problem of the contamination by the Goldstone boson [22].

<sup>3</sup>Besides the values of the Wilson hopping parameters corresponding to the light quark masses that we already listed in tab 1, for this computation we also use the quark propagator with  $\kappa_q = 0.1333$ .

<sup>4</sup>The explicit form of the function  $c^{\text{RI/MOM}}(\mu)$  is also given in Appendix of the present letter.

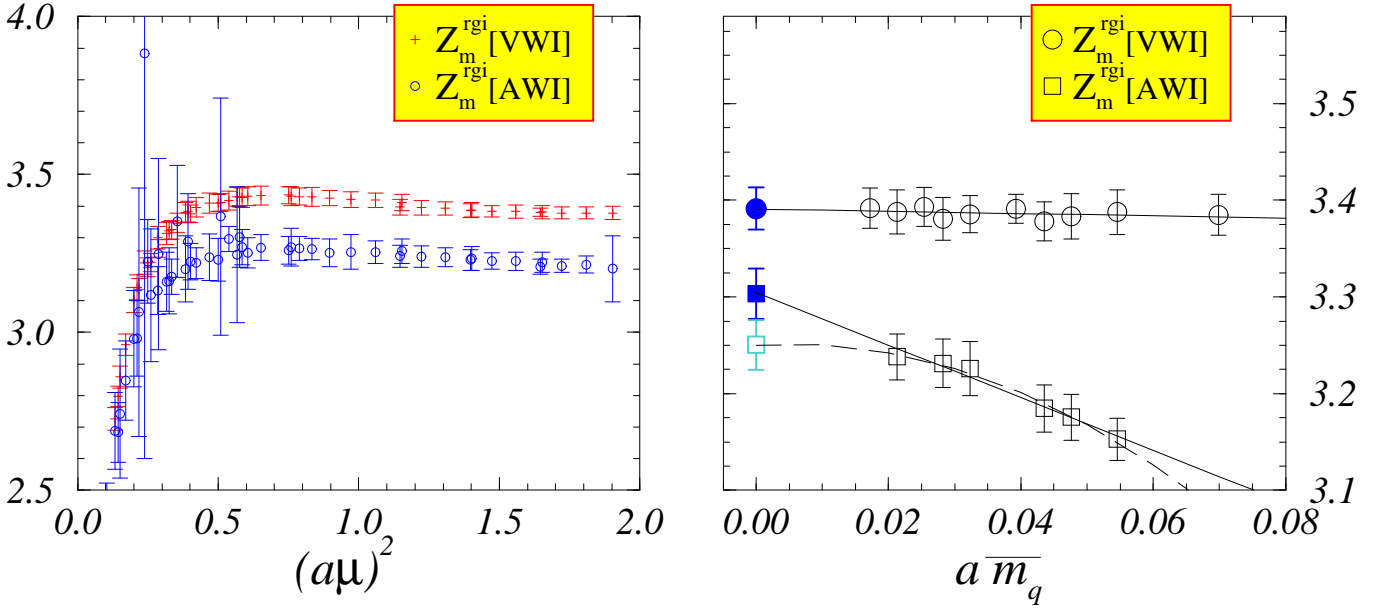


Figure 1: Mass renormalization constant: In the left figure we show the renormalization constants divided by the perturbative 4-loop anomalous dimension for the case of VWI ( $Z_m = 1/Z_S$ ) and AWI ( $\overline{Z}_m = Z_A/Z_P$ ) for a specific combination of  $\kappa_1 = 0.1349$  and  $\kappa_2 = 0.1352$ . On the right figure we show the extrapolation of the mass renormalization constants to the chiral limit. Dashed line depicts the quadratic extrapolation to  $\overline{Z}_m^{(0)\text{rgi}}$ .

The renormalization condition, by which this task is achieved for the mass renormalization constant, can be simply written as

$$\frac{Z_A^{(0)}}{Z_P^{(0)}}(\mu) = \lim_{\kappa_{1,2} \rightarrow \kappa_{cr}} \frac{Z_A}{Z_P}(\kappa_1, \kappa_2; \mu) = \lim_{\kappa_{1,2} \rightarrow \kappa_{cr}} \frac{m_1^{(\text{VWI})} \Gamma_P(\kappa_1; p) - m_2^{(\text{VWI})} \Gamma_P(\kappa_2; p)}{(m_1^{(\text{VWI})} - m_2^{(\text{VWI})}) \Gamma_A(\kappa_1, \kappa_2; p)} \Big|_{p^2 = \mu^2}. \quad (11)$$

Obviously, we can use only the non-degenerate quark mass combinations which, for our 4 values of  $\kappa_q$ , means 6 combinations. As in the previous case, we convert our result from the RI/MOM to the renormalization group invariant constant (at 4-loop level) and fit in the same window as before,  $1.0 \leq (\mu a)^2 \leq 1.8$  (see fig. 1(left)). With such extracted values for  $\overline{Z}_m^{\text{rgi}}$ , for each combination of the Wilson hopping parameters  $(\kappa_1, \kappa_2)$ , we then extrapolate to the chiral limit. This is also illustrated in fig. 1(right). Contrary to the first case, the mass dependence of the  $\overline{Z}_m^{\text{rgi}}$  is more pronounced. We extrapolate to the chiral limit linearly (filled square in fig. 1) to get our central value. In addition, we perform the quadratic extrapolation (the result of which is depicted by an empty square in fig. 1), and the difference between this and the central value is incorporated in the systematic uncertainty. Our result is

$$\overline{Z}_m^{(0)\text{rgi}} = 3.303(26)^{+0.000}_{-0.051}, \quad (12)$$

which in the  $\overline{\text{MS}}$  scheme reads

$$\overline{Z}_m^{(0)\overline{\text{MS}}}(2 \text{ GeV}) = 1.297(10)^{+0.000}_{-0.020}. \quad (13)$$

This result agrees well with the one of ref. [21],  $\overline{Z}_m^{(0)\overline{\text{MS}}}(2 \text{ GeV}) = 1.316(14)(17)$ .

## 2.3 Putting it all together

Now we combine all the results from tab. 1 with the renormalization constants discussed in the previous section to get the renormalization group invariant quark masses. At this stage (after including the renormalization constants) one can identify the results of the axial Ward identity as the sum of the heavy and the light quark mass. Since we also computed the light quark mass separately, we can now simply subtract it from the sum and work with the heavies only. The results are presented in tab. 2. In the same table we give the values of the heavy-light mesons for which the light quark mass has been interpolated to the light  $s$ -quark in a usual way (see *e.g.* ref. [23]). The mass of the  $D_s$  meson in lattice units is  $M_{D_s} \simeq 0.73(3)$ , where we use  $a^{-1}(m_{K^*}) = 2.7(1) \text{ GeV}$ . Thus the charm quark mass is to be found through an interpolation between the results for  $\kappa_Q = 0.125$  and  $\kappa_Q = 0.122$ . Notice

$\diamond$	$24^3 \times 64$			$24^3 \times 48$		
$\kappa_Q$	$m_Q^{\text{rgi}} [\text{AWI}]$	$m_Q^{\text{rgi}} [\text{VWI}]$	$M_{P_s}$	$m_Q^{\text{rgi}} [\text{AWI}]$	$m_Q^{\text{rgi}} [\text{VWI}]$	$M_{P_s}$
0.125	1.032(13)	0.868(3)	0.676(3)	1.000(14)	0.864(2)	0.681(5)
0.122	1.334(16)	1.013(2)	0.771(3)	1.306(16)	1.011(1)	0.777(4)
0.119	1.638(20)	1.141(1)	0.861(3)	1.592(19)	1.139(1)	0.866(4)

Table 2: Renormalized heavy quark masses directly accessed from our lattice.

that in tab. 2 we give also the results for a larger quark mass corresponding to  $\kappa_Q = 0.119$ . Although this value is not necessary for our final result for the charm quark mass, it will be helpful in assessing the amount of the systematic uncertainties which will be discussed in the next subsection.

Now, to get the value of the charm quark mass, we need to interpolate in the heavy meson mass to  $M_{D_s}$  and then simply read off the charm quark mass. To do so we need to choose an interpolating formula. We consider the following ones:

- (i)  $M_{P_s} = a_0 + a_1 m_Q + a_2 m_Q^2$ ;
- (ii)  $M_{P_s} = b_0 + b_1/m_Q + b_2/m_Q^2$ ;
- (iii)  $M_{P_s} = c_0 + c_1 m_Q + c_2/m_Q$ .

The first (i) is the naive linear interpolation ( $a_2 = 0$ ), the second (ii) comes from the heavy quark expansion and the third (iii) is the hybrid of the two. For (iii) we obviously need at least three points, and thus we have to use also the heaviest of our quarks from tab. 2. The complete situation is presented in tab. 3. Since the heavier quark is more prone to the  $\mathcal{O}((am)^2)$  artifacts, we first concentrate our discussion on the results of the first two interpolations in which we set  $a_2 = b_2 = 0$ . The results of the two interpolation formulae are totally consistent with each other, and we choose to quote the first one as our central number.

	$24^3 \times 64$ (Lattice-I)		$24^3 \times 48$ (Lattice-II)	
$\kappa_Q$	$m_c^{\text{rgi}}$ [AWI]	$m_c^{\text{rgi}}$ [VWI]	$m_c^{\text{rgi}}$ [AWI]	$m_c^{\text{rgi}}$ [VWI]
Form (i) ( $a_2 = 0$ )	1.190(84)	0.944(39)	1.136(100)	0.929(47)
Form (ii) ( $b_2 = 0$ )	1.171(83)	0.939(39)	1.126(98)	0.923(46)
Form (i)	1.186(84)	0.945(39)	1.135(100)	0.930(47)
Form (ii)	1.191(82)	0.947(39)	1.143(98)	0.932(46)
Form (iii)	1.187(84)	0.954(39)	1.137(100)	0.931(47)

Table 3: Charm quark mass in lattice units as obtained by using the interpolating formulae (i), (ii) and (iii), as discussed in the text. The first two are obtained without using the heaviest of our quarks (the one corresponding to  $\kappa_Q = 0.119$ ).

To get the final result, that can be confronted to the results of other approaches, we need to convert our values to the  $\overline{\text{MS}}$  scheme and express it in the physical units by using  $a^{-1}(m_{K^*}) = 2.7(1)$  GeV. After recalling that  $m_c^{\text{rgi}} \cdot c^{\overline{\text{MS}}}(\mu) = m_c^{\overline{\text{MS}}}(\mu)$ , where the function  $c^{\overline{\text{MS}}}(\mu)$  is known to 4-loop accuracy [24] (see App. of the present letter), one can easily solve that equation to obtain the standard value  $m_c^{\overline{\text{MS}}}(m_c)$ . Our results are:

$$\begin{aligned} \text{Lattice - I} \quad m_c^{\overline{\text{MS}}}(m_c)_{\text{VWI}} &= 1.144(3) \text{ GeV} , \\ m_c^{\overline{\text{MS}}}(m_c)_{\text{AWI}} &= 1.373(34) \text{ GeV} , \end{aligned} \quad (14)$$

$$\begin{aligned} \text{Lattice - II} \quad m_c^{\overline{\text{MS}}}(m_c)_{\text{VWI}} &= 1.132(3) \text{ GeV} , \\ m_c^{\overline{\text{MS}}}(m_c)_{\text{AWI}} &= 1.325(43) \text{ GeV} . \end{aligned} \quad (15)$$

The above results are obtained by using  $n_F = 0$  and  $\Lambda_{\text{QCD}}^{n_F=0} = 0.25$  GeV. We see that for both sets of our lattice data the two equivalent methods (VWI and AWI) give different results. The reason for that discrepancy most probably comes from the lattice artifacts which are  $\propto (am_Q)^n$  ( $n \geq 2$ ). One way of seeing that is to include the higher order effects at tree-level by employing the so-called EKLM factors [25] (see also the discussion in [26]). The leading effect of the EKLM factors to our result is  $\propto (am_Q)^2$  and it modifies the RGI charm quark mass as follows

$$\begin{aligned} m_c^{\text{rgi}}[\text{VWI}] &\longrightarrow \left[ 1 + \frac{(m_c^{(\text{VWI})})^2}{12} \right] m_c^{\text{rgi}}[\text{VWI}] , \\ m_c^{\text{rgi}}[\text{AWI}] &\longrightarrow \left[ 1 - \frac{(m_c^{(\text{VWI})})^2}{6} \right] m_c^{\text{rgi}}[\text{AWI}] , \end{aligned} \quad (16)$$

where  $m_c^{(\text{VWI})}$  stands for the bare lattice charm quark mass whose value is  $m_c^{(\text{VWI})} = 0.375(25)$ . When the above modification is included and we pass onto  $m_c^{\overline{\text{MS}}}(m_c)$ , from



our Lattice-I simulation, we get

$$\begin{aligned} m_c^{\overline{\text{MS}}}(m_c)_{\text{VWI}} &= 1.161(3) \text{ GeV}, \\ m_c^{\overline{\text{MS}}}(m_c)_{\text{AWI}} &= 1.342(33) \text{ GeV}. \end{aligned} \quad (17)$$

After comparing these to the results (14), we see that the inclusion of the tree level  $\mathcal{O}((am_Q)^2)$  effects makes our two results getting closer to each other, although it is not sufficient to remove the bulk of  $\mathcal{O}(a^2)$  corrections.

## 2.4 Systematic uncertainties and our final result

We will now briefly summarize the sources of systematic uncertainties and comment each one of them.

- The most important source of the systematic error are the lattice artifacts of  $\mathcal{O}((am)^n)$  ( $n \geq 2$ ). It is therefore highly important to repeat our calculation on the lattice with a smaller lattice spacing. As our central result we will quote the average of the two methods (VWI and AWI), for both our sets of data as given in eqs. (14,15). The larger statistical error will be attributed to our final result while the difference between any of the two methods and the averaged one is included in the systematic uncertainty. In other words we have:

$$\begin{aligned} \text{Lattice - I} \quad m_c^{\overline{\text{MS}}}(m_c) &= 1.26(3)(11) \text{ GeV}, \\ \text{Lattice - II} \quad m_c^{\overline{\text{MS}}}(m_c) &= 1.23(4)(10) \text{ GeV}. \end{aligned} \quad (18)$$

The results of our two simulations are very close to each other and for our final estimate of the charm quark mass we will quote the value obtained from the first lattice, which has the larger temporal extension.

- In our analysis, we used the value of the lattice spacing  $a^{-1} = 2.71(11) \text{ GeV}$ , as obtained from the comparison of the physical  $m_{K^*}$  and the one we computed on the lattice. We checked that if, instead of the above value, we use  $a^{-1} = 2.83(15)$  as obtained from the  $f_K$  decay constant, the final value of the charm quark gets larger by 2%.
- To be conservative, when taking the average of the masses obtained by using the two equivalent methods, we quoted the larger statistical error. However, the final result does not contain  $-1.5\%$  of the error on the renormalization constant  $\overline{Z}_m^{\text{rgi}}$  which we discussed in the text (see eq. (12)). Although the value of the renormalization constant  $Z_m^{\text{rgi}}$  does not suffer from the same uncertainty, to be on the safe side, we will add  $-1.5\%$  of error to our final result.
- Whenever needed, we used  $\Lambda_{\text{QCD}}^{n_F=0} = 0.25 \text{ GeV}$ . Varying this quantity by 10% allows one to cover all the presently available lattice estimates for  $\Lambda_{\text{QCD}}^{n_F=0}$  [20, 21]. The impact of that variation on the final charm quark mass value is  $\pm 1.6\%$ .

- In ref. [12], the authors also estimate the discretization errors on the improvement coefficient  $b_A - b_P$ . We have varied this coefficient by  $\pm 0.03$  which introduces the change in the central value for the charm quark mass by less than 1% (more precisely by  $\pm 0.6\%$ ).
- As mentioned in the previous subsection, the result obtained from the naive (linear) interpolation practically coincides with the one that we get when employing the interpolation motivated by the heavy quark expansion. Since we have the third (heavier) quark mass, we checked that the quadratic terms to both formulae (i,ii), or the use of the “hybrid” formula (iii) affects the value of the charm quark mass by no more than 1% (see tab. 3).
- For determination of the charm quark mass we used the masses of the heavy-light pseudoscalar mesons with the light quark mass (linearly) interpolated to the strange one. We checked that the quadratic interpolation in the light quark mass to get  $M_{P_s}$ , does not make any influence on our final charm quark mass value. In addition, we verified that our results remain unchanged if we use the vector heavy-light mesons instead of the pseudoscalars.

We sum the above sources of errors quadratically and obtain

$$m_c^{\overline{\text{MS}}}(m_c) = 1.26(3)(12) \text{ GeV} , \quad (19)$$

which is our final result. We note also that in passing from the RGI to the  $\overline{\text{MS}}$  value, we might have as well used  $n_F = 4$  and the value  $\alpha_s(m_\tau) = 0.334(22)$  [27]. This, however, does not modify the above result, namely we obtain  $m_c^{\overline{\text{MS}}}(m_c) = 1.27(3)(11) \text{ GeV}$ .

### 3 Summary and Perspectives

In this letter we have computed the charm quark mass on the lattice by taking the advantage of the full improvement of the Wilson action and operators by which all the lattice artifacts linear in lattice spacing are absent. Therefore, the computation of the charm quark mass on a reasonably fine grained lattice is expected to lead to the result close to the continuum limit. Results of our simulations, obtained at single value of the lattice spacing, indicate that the lattice artifacts  $\mathcal{O}(a^2)$  are sensible. As a consequence the two equivalent methods to compute the quark mass on the lattice yield different values. This fact largely dominates the systematic uncertainty of our calculation, and for the precise determination of the charm quark mass it is therefore important to go to ever smaller lattice spacings. Finer lattices, in turn, require larger lattices (to keep the same physical volume) and thus more powerful computing resources. By using APE1000 we plan to make such a study in the near future. We stress also that our computation has been performed in the quenched approximation. The (unknown) uncertainty introduced by quenching is not included in our systematic error estimate. A naïve guess points towards a decrease of the quenched value by  $\sim 5\%$ .<sup>5</sup> However we prefer to quote our result as the quenched one and wait for the partially unquenched computations to assess the amount of this source of systematic errors.

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<sup>5</sup> This “guesstimate” is based on the present observations according to which the unquenching of the light quark masses reduces their values by  $\sim 10\%$ , whereas  $m_b$  gets smaller by less than 2% [1].

# Appendix

The RGI quark mass is defined as

$$m^{\text{rgi}} = \frac{m^{\overline{\text{MS}}}(\mu)}{c^{\overline{\text{MS}}}(\mu)} = \frac{m^{\text{RI/MOM}}(\mu)}{c^{\text{RI/MOM}}(\mu)}, \quad (20)$$

where the functions  $c^{\overline{\text{MS}}}(\mu)$  and  $c^{\text{RI/MOM}}(\mu)$  are known to 4-loop accuracy [18, 24]. For  $n_F = 0$ , they read

$$c^{\overline{\text{MS}}}(\mu) = a_s(\mu)^{4/11} \left( 1 + 0.68733 a_s(\mu) + 1.51211 a_s(\mu)^2 + 4.05787 a_s(\mu)^3 \right), \quad (21)$$

$$c^{\text{RI/MOM}}(\mu) = a_s(\mu)^{4/11} \left( 1 + 2.02067 a_s(\mu) + 14.21925 a_s(\mu)^2 + 138.30689 a_s(\mu)^3 \right) \quad (22)$$

where, for short, we write  $a_s(\mu) \equiv \alpha_s(\mu)/\pi$ .

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